E. F. Bolinder

A. ANALYSIS OF BILATERAL, TWO-PORT NETWORKS BY THREE ARBITRARY IMPEDANCE OR REFLECTION-COEFFICIENT MEASUREMENTS

A general method of analyzing bilateral, two-port networks from three arbitrary impedance (admittance) or reflection-coefficient measurements, originally presented in the Quarterly Progress Report of April 15, 1956, page 126, has been developed and checked by numerical examples.

Geometrically, the method consists in stereographically mapping on the surface of the Riemann unit sphere three given output quantities and their corresponding measured input quantities. From the six points on the sphere, the fixed points and the multiplier of the normal (canonic) form of the linear fractional transformation that represents the network can be obtained by using Klein's three-dimensional generalization of the Pascal theorem.

The Pascal theorem states: "If a hexalateral is inscribed in a nondegenerate conic, the points of intersection of the pairs of opposite sides are collinear." In Fig. XX-1 the conic is a circle and the hexalateral is AB'CA'BC'A. The opposite sides $\overline{AB'}$ and $\overline{BA'}$ cut at P_1; $\overline{BC'}$ and $\overline{CB'}$ cut at P_2; and $\overline{AC'}$ and $\overline{CA'}$ cut at P_3. The three points P_1, P_2, and P_3 lie on a straight line, called the "Pascal line." By using the Cayley-Klein model of three-dimensional hyperbolic space, Klein was able to generalize the Pascal theorem to three dimensions. The generalized formulation (1) reads: "The non-Euclidean perpendiculars to opposite sides of a space hexalateral inscribed in a second-degree surface (for simplicity, the unit sphere) have a common non-Euclidean perpendicular." The theorem can be used for finding the fixed points and the multiplier of the normal form of the linear fractional transformation that represents the network.

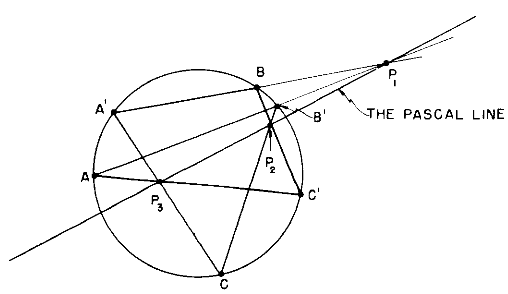

Fig. XX-1. Pascal's theorem.

The three sets of given output and measured input impedances and/or reflection coefficients are stereographically mapped on the Riemann unit sphere, so that the points A, B, C, A', B', and C' on the surface of the sphere are obtained, forming a space hexalateral AB'CA'BC'A. Then the non-Euclidean perpendicular to the opposite sides $\overline{AB'}$ and $\overline{BA'}$ is constructed. The procedure is repeated for $\overline{AC'}$ and $\overline{CA'}$ (or $\overline{BC'}$ and $\overline{CB'}$). Finally, the common perpendicular to the two constructed perpendiculars is obtained. The fixed points of the transformation are obtained as the points at which the common perpendicular cuts the surface of the sphere. The common perpendicular is called the "inner axis" of the transformation.

The multiplier of the transformation is obtained by selecting an arbitrary point E on the surface of the sphere and then connecting, for example, A and A' with E and constructing the non-Euclidean perpendiculars to $\overline{AE}$ and the inner axis, and to $\overline{A'E}$ and the inner axis. The hyperbolic distance between the two crossover points between the perpendiculars and the inner axis is λ', and the two planes through the perpendiculars and the inner axis make the non-Euclidean (elliptic) angle λ''. The quantities λ' and λ'' can be considered as forming a "complex angle," $-j\lambda = \lambda'' - j\lambda'$, or a "complex distance," $\lambda = \lambda' + j\lambda''$. This complex quantity, together with the fixed points, completely specifies the normal form of the linear fractional transformation

$$\left.\begin{aligned} \frac{Z' - Z_{f1}}{Z' - Z_{f2}} &= q\,\frac{Z - Z_{f1}}{Z - Z_{f2}} \\ q &= e^{2\lambda} = e^{2(\lambda' + j\lambda'')} \end{aligned}\right\} \tag{1}$$

where Z is the output impedance, Z' is the input impedance, Z_{f1} and Z_{f2} are the fixed points, and q is the multiplier. After Eq. 1 is known, simple calculations yield the equivalent T- or Π-networks of the analyzed network.

Analytically, all of the constructions of the geometric part can be performed by using the theory of invariance of quadratic forms and complex spherical trigonometry. Analytic expressions can be found for a line that cuts the sphere, for a line that is non-Euclidean perpendicular to two given lines that cut the sphere, and for the complex angle (distance) between two lines that cut the sphere, by using a theory discribed by Klein in his lectures on the hypergeometric function (2). Only the final expressions for the fixed points and the multiplier will be given here. The fixed points are expressed in terms of three given output impedances Z_A, Z_B, and Z_C, and three measured input impedances Z'_A, Z'_B, and Z'_C. They are

$$\left.\begin{matrix} Z_{f1} \\ Z_{f2} \end{matrix}\right\} = \frac{k_7 \pm \left(k_7^2 - 4k_8\right)^{1/2}}{2} \tag{2}$$

where

$$k_7 = \frac{k_1k_6 - k_3k_4}{k_1k_5 - k_2k_4} \quad \text{and} \quad k_8 = \frac{k_2k_6 - k_3k_5}{k_1k_5 - k_2k_4} \tag{3}$$

with

$$\left.\begin{aligned}
k_1 &= (Z_A - Z'_A) - (Z_B - Z'_B) \\
k_2 &= Z_A Z'_B - Z'_A Z_B \\
k_3 &= (Z_A - Z'_A)\, Z_B Z'_B - (Z_B - Z'_B)\, Z_A Z'_A \\
k_4 &= (Z_A - Z'_A) - (Z_C - Z'_C) \\
k_5 &= Z_A Z'_C - Z'_A Z_C \\
k_6 &= (Z_A - Z'_A)\, Z_C Z'_C - (Z_C - Z'_C)\, Z_A Z'_A
\end{aligned}\right\} \tag{4}$$

The multiplier is expressed in terms of the impedances Z_A and Z'_A, the fixed points Z_{f1} and Z_{f2}, and the arbitrary impedance Z_E. Thus

$$q = e^{2\lambda} \tag{5}$$

where

$$\lambda = \lambda' + j\lambda'' = -\cosh^{-1} \frac{D_{12}}{\sqrt{D_{11}}\sqrt{D_{22}}} \tag{6}$$

with

$$\left.\begin{aligned}
D_{11} &= 4(k_{10} - k_9 k_{11}) \\
D_{22} &= 4(k_{13} - k_{12} k_{14}) \\
D_{12} &= 2(2k_{10}k_{13} - k_9 k_{14} - k_{11} k_{12})
\end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned}
k_9 &= (Z_A + Z_E) - (Z_{f1} - Z_{f2}) \\
k_{10} &= Z_A Z_E - Z_{f1} Z_{f2} \\
k_{11} &= (Z_{f1} + Z_{f2})\, Z_A Z_E - (Z_A + Z_E)\, Z_{f1} Z_{f2} \\
k_{12} &= (Z'_A + Z_E) - (Z_{f1} - Z_{f2}) \\
k_{13} &= Z'_A Z_E - Z_{f1} Z_{f2} \\
k_{14} &= (Z_{f1} + Z_{f2})\, Z'_A Z_E - (Z'_A + Z_E)\, Z_{f1} Z_{f2}
\end{aligned}\right\} \tag{8}$$

The geometric-analytic method presented here has been checked by calculation of three numerical examples, consisting of a pure resistive network (attenuator), a pure reactive network (lowpass network), and an RLC network. It has also been compared with a pure analytic method obtained by inserting the given and measured values directly into the linear fractional transformation. While the amount of work involved in calculating the numerical examples is roughly the same by both methods, the new method has the advantage of giving a visual geometric picture of the different operations of the method.

The complete theory of the geometric-analytic method will be published in Ericsson Technics, Stockholm, Sweden.

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B. GEOMETRIC-ANALYTIC THEORY OF NOISY TWO-PORTS

It has been shown in Section XX-A, and in previous reports (1), how the Cayley-Klein model of three-dimensional hyperbolic space can be used constructively for impedance transformations through bilateral, two-port networks. A generalization of this theory to a geometric-analytic theory of noisy two-ports will now be outlined.

The input voltage V' and the current I' are linearly related to the output voltage V and the current I of a two-port network.

$$\psi' = \begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = T\psi \tag{1}$$

The input impedance $Z' = V'/I'$ is expressed in terms of the output impedance $Z = V/I$ by the linear fractional transformation

$$Z' = \frac{aZ + b}{cZ + d} \tag{2}$$

For noisy two-port networks, we exchange Eqs. 1 and 2 for

$$Q' = \begin{pmatrix} Q'_1 \\ Q'_2 \\ Q'_3 \\ Q'_4 \end{pmatrix} = \begin{pmatrix} \overline{V'V'^*} \\ \overline{V'I'^*} \\ \overline{V'^*I'} \\ \overline{I'I'^*} \end{pmatrix} = \begin{pmatrix} aa^* & ab^* & ba^* & bb^* \\ ac^* & ad^* & bc^* & bd^* \\ ca^* & cb^* & da^* & db^* \\ cc^* & cd^* & dc^* & dd^* \end{pmatrix} \begin{pmatrix} \overline{VV^*} \\ \overline{VI^*} \\ \overline{V^*I} \\ \overline{II^*} \end{pmatrix} = LQ \tag{3}$$

and

$$s'^2 = \frac{Q'_1}{Q'_4} = \frac{aa^* s^2 + ab^* \zeta + ba^* \zeta^* + bb^*}{cc^* s^2 + cd^* \zeta + dc^* \zeta^* + dd^*} \tag{4a}$$

$$\zeta' = \frac{Q'_2}{Q'_4} = \frac{ac^* s^2 + ad^* \zeta + bc^* \zeta^* + bd^*}{cc^* s^2 + cd^* \zeta + dc^* \zeta^* + dd^*} \tag{4b}$$

$$\zeta'^* = \frac{Q'_3}{Q'_4} = \frac{ca^* s^2 + cb^* \zeta + da^* \zeta^* + db^*}{cc^* s^2 + cd^* \zeta + dc^* \zeta^* + dd^*} \tag{4c}$$

A star indicates a complex-conjugate quantity and a bar indicates an ensemble average. With complete correlation, $s^2 = \zeta\zeta^*$, and Eq. 4a reduces to $s'^2 = \zeta'\zeta'^*$; Eq. 4b reduces to Eq. 2; and Eq. 4c reduces to the complex conjugate of Eq. 2.

If we set

$$\left.\begin{aligned} P_1 &= \frac{1}{2}(Q_2 + Q_3) = \frac{1}{2}(\overline{VI^*} + \overline{V^*I}) \\ P_2 &= -\frac{j}{2}(Q_2 - Q_3) = -\frac{j}{2}(\overline{VI^*} - \overline{V^*I}) \\ P_3 &= \frac{1}{2}(Q_1 - Q_4) = \frac{1}{2}(\overline{VV^*} - \overline{II^*}) \\ P_4 &= \frac{1}{2}(Q_1 + Q_4) = \frac{1}{2}(\overline{VV^*} + \overline{II^*}) \end{aligned}\right\} \tag{5}$$

then Eqs. 3 and 4 transform into

$$P' = \begin{pmatrix} P'_1 \\ P'_2 \\ P'_3 \\ P'_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = MP \tag{6}$$

and

$$\left.\begin{aligned} x' &= \frac{P_1'}{P_4'} = \frac{a_1x + a_2y + a_3z + a_4}{d_1x + d_2y + d_3z + d_4} \\ y' &= \frac{P_2'}{P_4'} = \frac{b_1x + b_2y + b_3z + b_4}{d_1x + d_2y + d_3z + d_4} \\ z' &= \frac{P_3'}{P_4'} = \frac{c_1x + c_2y + c_3z + c_4}{d_1x + d_2y + d_3z + d_4} \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} a_1 &= \mathrm{Re}(ad^* + bc^*) & c_1 &= \mathrm{Re}(ab^* - cd^*) \\ a_2 &= \mathrm{Im}(bc^* + da^*) & c_2 &= \mathrm{Im}(ba^* + cd^*) \\ a_3 &= \mathrm{Re}(ac^* - bd^*) & c_3 &= \tfrac{1}{2}(|a|^2 - |b|^2 - |c|^2 + |d|^2) \\ a_4 &= \mathrm{Re}(ac^* + bd^*) & c_4 &= \tfrac{1}{2}(|a|^2 + |b|^2 - |c|^2 - |d|^2) \\ b_1 &= \mathrm{Im}(ad^* + bc^*) & d_1 &= \mathrm{Re}(ab^* + cd^*) \\ b_2 &= \mathrm{Re}(ad^* - bc^*) & d_2 &= \mathrm{Im}(ba^* - cd^*) \\ b_3 &= \mathrm{Im}(ac^* - bd^*) & d_3 &= \tfrac{1}{2}(|a|^2 - |b|^2 + |c|^2 - |d|^2) \\ b_4 &= \mathrm{Im}(ac^* + bd^*) & d_4 &= \tfrac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2) \end{aligned}\right\} \tag{8}$$

Thus, instead of Eqs. 1 and 2 which represent noise-free, two-port networks, we use Eqs. 3 and 4 or Eqs. 6 and 7 to represent noisy two-ports. The complex vector ψ in Eq. 1 is analogous in optics to the complex Maxwell vector; the complex four-vector Q in Eq. 3 is analogous to the complex Stokes vector; and the real four-vector P in Eq. 6 is analogous to the real Stokes vector (2). The complex 2×2 matrix T in Eq. 1 is analogous in optics to the Jones matrix; the complex 4×4 matrix L in Eq. 3 and the real 4×4 matrix M in Eq. 6 are analogous to the 4×4 matrices used by Soleillet, Perrin, Mueller, Parke (3), and others. The matrix L is the Kronecker (tensor) product $T \times T^*$. A connecting link between the 2×2 and 4×4 matrix representations is Wiener's generalized harmonic analysis. The components of the Q-vector are the components of a 2×2 Hermitian coherency matrix (4).

For bilateral two-port networks, Eq. 4 can be interpreted geometrically as a movement in a Poincaré model of three-dimensional hyperbolic space that has the ζ-plane,

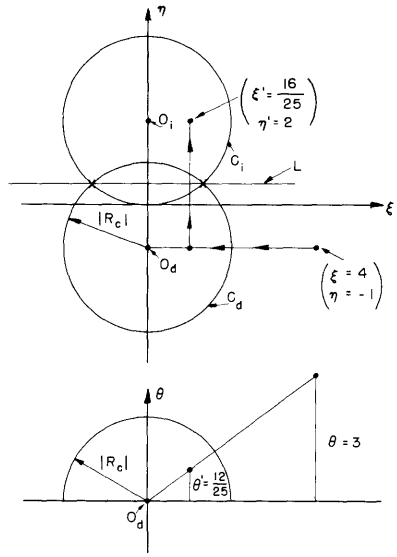

Fig. XX-2. Example of transformation by the generalized isometric circle method.

$\zeta = \xi + j\eta$, as the absolute surface. The transformation of a point (ξ, η, θ) into (ξ', η', θ') can be performed geometrically by a generalized form of the isometric circle method (5), in which circles in the ζ-plane are extended to hemispheres. The third coordinate, θ, is connected to ζ and s by the equation

$$s^2 = |\zeta|^2 + \theta^2 = \xi^2 + \eta^2 + \theta^2 \tag{9}$$

A simple example is shown in Fig. XX-2. The point $(\xi, \eta, \theta) = (4, -1, 3)$ is transformed through a lossless, bilateral, two-port network $(a = 1,\ b = -j,\ c = -j/2,\ d = 1/2)$ to $(\xi', \eta', \theta') = (16/25, 2, 12/25)$.

Equation 7 can be interpreted geometrically as a movement in a Cayley-Klein model of three-dimensional hyperbolic space that has the unit sphere as the absolute surface. For noisy two-ports the point (x, y, z) is situated inside the sphere. It is transformed into (x', y', z') by geometric-analytic methods (1). The connections between the ξ, η, and θ coordinates and the x, y, and z coordinates are

$$\left.\begin{aligned} \zeta &= \xi + j\eta = \frac{x + jy}{1 - z} \\ \theta &= \frac{\sqrt{1 - x^2 - y^2 - z^2}}{1 - z} \\ s^2 &= \frac{1 + z}{1 - z} \end{aligned}\right\} \tag{10}$$

The complex variable ζ, which can be written

$$\zeta = \frac{Q_2}{Q_4} = \frac{P_1 + jP_2}{P_4 - P_3} \tag{11}$$

can be considered as a complex correlation impedance.

This presentation of the geometric-analytic theory shows that, from the standpoint of projective geometry, lossless, bilateral, two-port networks are essentially one-dimensional in structure and can be described by real numbers; lossy, bilateral, two-port networks are essentially two-dimensional and can be described by complex numbers; noisy, two-port networks are, on an "impedance" basis, essentially three-dimensional and can be described by combinations made up of a complex number and a real number; and, finally, noisy, two-port networks are, on a power basis, essentially four-dimensional and can be described by combinations made up of a complex number and two real numbers, or of four real numbers.

It is interesting to compare the geometric-analytic theory with a theory of noisy two-port networks of Rothe and Dahlke (6), and a theory of noise in longitudinal electron beams of Haus (7, 8), and of Haus and Robinson (9).

Rothe and Dahlke split the noisy network into a noisy part and a noise-free part (see Fig. XX-3). In Fig. XX-3, r_n indicates an equivalent noise resistance, g_n an equivalent noise conductance, and Z_{cor} a complex correlation impedance. Expressed in terms of these variables (and with the Boltzmann constant k, the temperature T_o, and the bandwidth Δf) the Q-vector is

$$Q = 4kT_o \Delta f\, g_n \begin{pmatrix} \dfrac{r_n}{g_n} + |Z_{cor}|^2 \\ Z_{cor} \\ Z_{cor}^* \\ 1 \end{pmatrix} \tag{12}$$

Thus, we have

$$\left.\begin{aligned} s^2 &= \frac{r_n}{g_n} + |Z_{cor}|^2 \\ \xi + j\eta &= \zeta = Z_{cor} = R_{cor} + jX_{cor} \\ \theta &= \sqrt{\frac{r_n}{g_n}} \end{aligned}\right\} \tag{13}$$

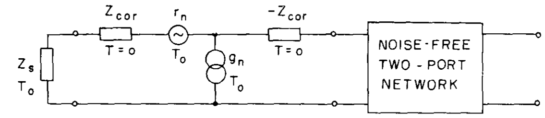

Fig. XX-3. Splitting of a noisy, two-port network into noisy and noise-free parts, according to Rothe and Dahlke.

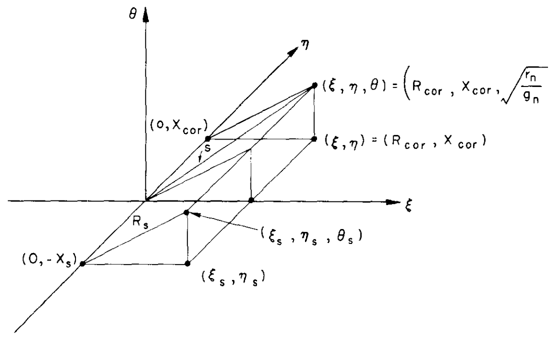

Fig. XX-4. Noise tuning and noise matching.

The important, always positive, quantity, $Q_1Q_4 - Q_2Q_3 = P_4^2 - P_3^2 - P_2^2 - P_1^2$, which for noise-free networks is zero, for noisy, two-port networks is $(4kT_o\Delta f)^2 r_n g_n$.

If a signal source with impedance $Z_s = R_s + jX_s$ is connected to the input of the network (see Fig. XX-3), then noise tuning and noise matching (6) is obtained when $(\zeta, \eta, \theta) = (\zeta_s, -\eta_s, \theta_s)$, so that

$$\left.\begin{aligned} X_s &= -X_{cor} \\ R_s^2 &= R_{cor}^2 + \frac{r_n}{g_n} \end{aligned}\right\} \tag{14}$$

Equations 14 constitute a generalization of ordinary tuning and matching to three dimensions. See Fig. XX-4.

By using Eqs. 3 and 4, the transformation formulas for r_n, g_n, and Z_{cor} deduced by Dahlke (10) are immediately found.

In the theory of longitudinal electron beams Haus defines the self-power density spectrum (SPDS) of the kinetic noise-voltage modulation Φ; the SPDS of the noise-current modulation Ψ; and the cross-power density spectrum (CPDS) between the kinetic voltage and current modulations θ. These variables can be combined to form a Q-vector

$$Q = 4\pi\Delta f \begin{pmatrix} \Phi \\ \theta \\ \theta^* \\ \Psi \end{pmatrix} \tag{15}$$

The application of the geometric-analytic theory to network theory, electron-beam theory, antenna theory, acoustics, optics, and quantum mechanics will be the subject of further research.

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